CHENNAI Mathematical Institute

MSc Applications of Mathematics

Entrance Examination, 2010

Part A

For each of the statements given below, state whether it is True or False and give brief reasons in the sheets provided. Marks will be given only when reasons are provided.

1. Let $5 \leq k < n$. Then $2k$ divides $n(n-1)\dots(n-k+1)$.
2. Let $p(x)$ be a polynomial of degree n with distinct real roots $x_1, x_2, \dots x_n$. Then $p'(x_1) \neq 0$. (Here, p' denotes the derivative of p.)
3. For all $x < 0$
$$e^x(1-e^x) \leq \frac{1}{4}.$$
4. For a positive integer $n > 1$,
$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$
5. Let $f(x) = |x-1| + |x-2|^3$ for $x \in \mathbb{R}$. Then f is not differntiable at $x = 1$ and $x = 2$ and is differentiable at all other points.
6. Let $\{a_n\}, \{b_n\}$ be sequences of strictly positive real numbers such that
$$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = b.$$
Let $c_n = a_n b_n$ and $d_n = \frac{a_n}{b_n}$. Then the sequences $\{c_n\}, \{d_n\}$ also converge to zero.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+1) = f(x)$, $\forall x$. Then f attains its supremum.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq |\sin(x) - \sin(y)|$, $\forall x, y$. Then f is continuous.
9. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$- matrices each having rank n. Then rank of $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is also n.
10. Let $\mathbf{A}$ be a $n \times n$-matrix such that the equation $\mathbf{Ax} = \mathbf{0}$ has a non-zero solution. Then there exists a vector $\mathbf{y}$ such that the equation $\mathbf{Ax} = \mathbf{y}$ has no solution.

Part B

Answer all questions.

1. Let $p(x) = x^2 + 2bx + c$ be a quadratic form, where b, c are real numbers. If $b^2 < c$ show that $p(x) > 0$ for all x. Is the converse true? Give reasons for your answer.

2. For $0 \leq x \leq \frac{\pi}{4}$, let $f(x) = \sin(x) + \cos(x)$. Find the infimum of $f(x)$ over the interval $x \in [0, \frac{\pi}{4}]$.

3. Let $A = (a_{ij})$ be a $n \times n$ matrix where $a_{ij} = (i + j)$. Find the rank of A.

4. Let f be a function on $\mathbb{R}$ such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$ where f' denotes the derivative of f and $f(0) = 1$. Show that $f(x) = \exp(x)$ for all x.

5. Show that f defined below is continuous and differentiable at $x = 0$.

$$f(x) = x^2 \sin(\frac{1}{x^2}) \text{ for } x \neq 0$$

   and $f(0) = 0$.

6. Let sequences $\{a_n\}$ and $\{b_n\}$ be defined via $a_n = 2^{-n}$ and $b_n = 3^{-n}$, $n \geq 1$. Let $c_n = \frac{a_n}{b_n}$ and $d_n = \frac{b_n}{a_n}$ for $n \geq 1$. Do the sequences $\{c_n\}$, $\{d_n\}$ converge as n tends to ∞? Justify your answer.

7. Suppose the series $\sum b_n$ is conditionally convergent but not absolutely convergent. What is the radius of convergence of the power series $p(x) = \sum b_n x^n$. Give reasons.

8. Suppose 5 boys and 4 girls are to be arranged in a queue such that between any two boys there is at least one girl. Find the number of such arrangements possible.

9. Let $\mathbf{A}$ be a real symmetric non-negative difinite $n \times n$ matrix. Let $\mathbf{B} = \mathbf{I} + \mathbf{A}$ where I denotes $n \times n$ identity matrix. Show that $\mathbf{B}$ is positive definite and non-singular.

10. Give an example of a sequence $\{a_n\}$ such that a_n does not converge but $b_n = \frac{a_1 + \ldots + a_n}{n}$ converges.